

\mathbb{B} -valued monogenic functions and their applications to boundary value problems in displacements of 2-D Elasticity

S.V. Gryshchuk

Institute of Mathematics,

National Academy of Sciences of Ukraine,

Tereshchenkivska Str. 3, 01601, Kyiv, Ukraine

gryshchuk@imath.kiev.ua, serhii.gryshchuk@gmail.com

January 8, 2016

Abstract

Consider the commutative algebra \mathbb{B} over the field of complex numbers with the bases $\{e_1, e_2\}$ such that $(e_1^2 + e_2^2)^2 = 0$, $e_1^2 + e_2^2 \neq 0$. Let D be a domain in xOy , $D_\zeta := \{xe_1 + ye_2 : (x, y) \in D\} \subset \mathbb{B}$. We say that \mathbb{B} -valued function $\Phi: D_\zeta \rightarrow \mathbb{B}$, $\Phi(\zeta) = U_1 e_1 + U_2 i e_1 + U_3 e_2 + U_4 i e_2$, $\zeta = xe_1 + ye_2$, $U_k = U_k(x, y): D \rightarrow \mathbb{R}$, $k = \overline{1, 4}$, is *monogenic* in D_ζ iff Φ has the classic derivative in every point in D_ζ . Every U_k , $k = \overline{1, 4}$, is a biharmonic function in D . A problem on finding an elastic equilibrium for isotropic body D by given boundary values on ∂D of partial derivatives $\frac{\partial u}{\partial v}$, $\frac{\partial v}{\partial y}$ for displacements u, v is equivalent to BVP for monogenic functions, which is to find Φ by given boundary values of U_1 and U_4 .

AMS Subject Classification (2010): Primary 30G35, 31A30; Secondary 74B05

Key Words and Phrases: biharmonic equation, biharmonic algebra, biharmonic plane, monogenic function of the biharmonic plane, Schwarz-type boundary value problem, isotropic plane strain, elastic equilibrium.

1 Monogenic functions in the biharmonic algebra associated with the biharmonic equation

We say that an associative commutative two-dimensional algebra \mathbb{B} with the unit 1 over the field of complex numbers \mathbb{C} is *biharmonic* if in \mathbb{B} there exists a *biharmonic* basis, i.e., a bases $\{e_1, e_2\}$ satisfying the conditions

$$(e_1^2 + e_2^2)^2 = 0, \quad e_1^2 + e_2^2 \neq 0. \quad (1)$$

V. F. Kovalev and I. P. Mel'nichenko [1] found a multiplication table for a biharmonic basis $\{e_1, e_2\}$:

$$e_1 = 1, \quad e_2^2 = e_1 + 2ie_2, \quad (2)$$

where i is the imaginary complex unit.

In [2], I. P. Mel'nichenko proved that there exists the unique biharmonic algebra \mathbb{B} and he constructed all biharmonic bases.

Consider a *biharmonic plane* $\mu := \{\zeta = xe_1 + ye_2 : x, y \in \mathbb{R}\}$ which is a linear span of the elements e_1, e_2 of the biharmonic basis (2) over the field of real numbers \mathbb{R} . With a domain D of the Cartesian plane xOy we associate the congruent domain $D_\zeta := \{\zeta = xe_1 + ye_2 : (x, y) \in D\}$ in the biharmonic plane μ , and corresponding domain in the complex plane \mathbb{C} : $D_z := \{z = x + iy : (x, y) \in D\}$.

Let D_* be a domain in xOy or in μ . Denote by ∂D_* a boundary of a domain D_* , $\text{cl}D_*$ means a closure of a domain D_* .

In what follows, $(x, y) \in D$, $\zeta = xe_1 + ye_2 \in D_\zeta$, $z = x + iy \in D_z$.

Inasmuch as divisors of zero don't belong to the biharmonic plane, one can define the derivative $\Phi'(\zeta)$ of function $\Phi: D_\zeta \rightarrow \mathbb{B}$ in the same way as in the complex plane:

$$\Phi'(\zeta) := \lim_{h \rightarrow 0, h \in \mu} (\Phi(\zeta + h) - \Phi(\zeta)) h^{-1}.$$

We say that a function $\Phi: D_\zeta \rightarrow \mathbb{B}$ is *monogenic* in a domain D_ζ and, denote by $\Phi \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$, iff the derivative $\Phi'(\zeta)$ exists in every point $\zeta \in D_\zeta$.

Every function $\Phi: D_\zeta \rightarrow \mathbb{B}$ has a form

$$\Phi(\zeta) = U_1(x, y) e_1 + U_2(x, y) ie_1 + U_3(x, y) e_2 + U_4(x, y) ie_2, \quad (3)$$

where $\zeta = xe_1 + ye_2$, $U_k: D \rightarrow \mathbb{R}$, $k = \overline{1, 4}$.

Every real component U_k , $k = \overline{1, 4}$, in expansion (3) we denote by $U_k[\Phi]$, i.e., for $k \in \{1, \dots, 4\}$: $U_k[\Phi(\zeta)] := U_k(x, y)$ for all $\zeta = xe_1 + ye_2 \in D_\zeta$.

It is established in the paper [1] that a function $\Phi: D_\zeta \rightarrow \mathbb{B}$ is monogenic in a domain D_ζ if and only if components U_k , $k = \overline{1, 4}$, of the expression (3) are differentiable in the domain D and the following analog of the Cauchy – Riemann conditions is satisfied:

$$\frac{\partial \Phi(\zeta)}{\partial y} e_1 = \frac{\partial \Phi(\zeta)}{\partial x} e_2. \quad (4)$$

In an extended form the condition (4) for the monogenic function (3) is equivalent to the system of four equations (cf., e.g., [1, 3]) with respect to components $U_k = U_k[\Phi]$, $k = \overline{1, 4}$, in (3):

$$\frac{\partial U_1(x, y)}{\partial y} = \frac{\partial U_3(x, y)}{\partial x}, \quad (5)$$

$$\frac{\partial U_2(x, y)}{\partial y} = \frac{\partial U_4(x, y)}{\partial x}, \quad (6)$$

$$\frac{\partial U_3(x, y)}{\partial y} = \frac{\partial U_1(x, y)}{\partial x} - 2 \frac{\partial U_4(x, y)}{\partial x}, \quad (7)$$

$$\frac{\partial U_4(x, y)}{\partial y} = \frac{\partial U_2(x, y)}{\partial x} + 2 \frac{\partial U_3(x, y)}{\partial x}. \quad (8)$$

It is proved in the paper [1] that a function $\Phi(\zeta)$ having derivatives till fourth order in D_ζ satisfies the two-dimensional biharmonic equation

$$(\Delta_2)^2 U(x, y) := \left(\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) U(x, y) = 0 \quad (9)$$

in the domain D owing to the relations (1) and

$$(\Delta_2)^2 \Phi(\zeta) = \Phi^{(4)}(\zeta) (e_1^2 + e_2^2)^2.$$

Therefore, every component $U_k: D \rightarrow \mathbb{R}$, $k = \overline{1, 4}$, of the expansion (3) satisfies also the equation (9), i. e. U_k is a *biharmonic function* in the domain D .

It is proved [3] that a monogenic function $\Phi: D_\zeta \rightarrow \mathbb{B}$ has derivatives $\Phi^{(n)}(\zeta)$ of all orders in the domain D_ζ and, consequently, satisfies the two-dimensional biharmonic equation (9).

In the papers [3–6] it was also proved such a fact that every biharmonic function $U_1(x, y)$ in a bounded simply connected domain D is the first component of the expansion (3) of monogenic function $\Phi: D_\zeta \rightarrow \mathbb{B}$, moreover, all such functions Φ are found in an explicit form.

Basic analytic properties of monogenic functions in a biharmonic plane are similar to properties of holomorphic functions of the complex variable. More exactly, analogues of the Cauchy integral theorem and integral formula, the Morera theorem, the uniqueness theorem, the Taylor and Laurent expansions are established in papers [5–7].

2 Auxiliary statement

Theorem 1. *Let W be an arbitrary fixed biharmonic in a domain D function; $\Phi_* \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$, $\Phi \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$ and $U_1[\Phi_*] = W$, $\Phi := \Phi''_*$. Then the following formulas are true:*

$$\frac{\partial^2 W(x, y)}{\partial x^2} = U_1[\Phi(\zeta)], \quad \frac{\partial^2 W(x, y)}{\partial y^2} = U_1[\Phi(\zeta)] - 2U_4[\Phi(\zeta)], \quad (10)$$

for every $(x, y) \in D$, $\zeta = xe_1 + ye_2 \in D_\zeta$.

Proof. There exists $\Phi_* \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$ such that

$$U_1[\Phi_*(\zeta)] = W(x, y) \quad \forall \zeta \in D_\zeta. \quad (11)$$

A derivative of a function Φ_* is represented by the equality $\Phi'_* = \frac{\partial \Phi_*}{\partial x}$. Therefore, we have the following equalities

$$U_k[\Phi'_*(\zeta)] = \frac{\partial U_k[\Phi_*(\zeta)]}{\partial x}, \quad k \in \{1, \dots, 4\}, \quad \forall \zeta \in D_\zeta. \quad (12)$$

Using the equality (5) for monogenic function Φ_* , deliver the equality

$$\frac{\partial U_1[\Phi_*(\zeta)]}{\partial y} = \frac{\partial U_3[\Phi_*(\zeta)]}{\partial x} \quad \forall \zeta \in D_\zeta,$$

substituting into which successively equalities (12) with $k = 3$ and (11), as a result, obtain

$$U_3[\Phi'_*(\zeta)] = \frac{\partial W(x, y)}{\partial y} \quad \forall \zeta \in D_\zeta. \quad (13)$$

Now, substituting (11) in (12) with $k = 1$, obtain

$$U_1[\Phi'_*(\zeta)] = \frac{\partial W(x, y)}{\partial x} \quad \forall \zeta \in D_\zeta. \quad (14)$$

Since, $\Phi \equiv \Phi'' = \frac{\partial \Phi'_*}{\partial x}$, therefore, it implies the equalities

$$U_k[\Phi(\zeta)] = \frac{\partial U_k[\Phi'_*(\zeta)]}{\partial x}, \quad k \in \{1, \dots, 4\}, \quad \forall \zeta \in D_\zeta. \quad (15)$$

Substituting (14) in the equality (15) with $k = 1$, we get the first equality in (10).

Using the formula (7) for monogenic function Φ'_* , we have

$$\frac{\partial U_3[\Phi'_*(\zeta)]}{\partial y} = \frac{\partial U_1[\Phi'_*(\zeta)]}{\partial x} - 2 \frac{\partial U_4[\Phi'_*(\zeta)]}{\partial x} \quad \forall \zeta \in D_\zeta. \quad (16)$$

Finally, substituting in series relations (13), (15) with $k = 1$ and $k = 4$ into (16), obtain the second equality in (10). The theorem is proved.

3 Displacements-type problem

We shall assume further that D is a bounded simply connected domain in the Cartesian plane xOy . Let $\tau: D \rightarrow \mathbb{R}$ be a real-valued function, then by $\mathcal{C}^k(D)$, $k \in \{0, 1, \dots\}$, we denote a class of functions having continuous derivatives up to the k -th order inclusively, $\mathcal{C}(D) := \mathcal{C}^0(D)$. If $\tau \in \mathcal{C}^2(D)$ then $\Delta_2 \tau := \frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial y^2}$. For a function $\tau \in \mathcal{C}(D)$ denote by $\tau \in \mathcal{C}(\text{cl}D)$ if there exists a finite limit

$$\tau(x, y)|_{(x_0, y_0)} := \lim_{(x, y) \in D, (x, y) \rightarrow (x_0, y_0)} \tau(x, y) \quad \forall (x_0, y_0) \in \partial D.$$

Let a function Φ is of the type $\Phi: D_\zeta \longrightarrow \mathbb{B}$. Denote by $\Phi \in \mathcal{C}(\text{cl}D_\zeta)$ if and only if $U_k[\Phi] \in \mathcal{C}(\text{cl}D)$, $k \in \{1, \dots, 4\}$. In this case we use for every $\zeta_0 = x_0 e_1 + y_0 e_2 \in \partial D_\zeta$ notations

$$U_k[\Phi(\zeta_0)] := U_k \left[\lim_{\zeta \in D_\zeta, \zeta \rightarrow \zeta_0 \in \partial D_\zeta} \Phi(\zeta) \right], k = \overline{1, 4}.$$

Consider a boundary value problem: to find in D partial derivatives $\mathcal{V}_1 := \frac{\partial u}{\partial x}$, $\mathcal{V}_2 := \frac{\partial v}{\partial y}$ for displacements $u = u(x, y)$, $v = v(x, y)$ of an elastic isotropic body occupying D , when their boundary values are given on the boundary ∂D :

$$\mathcal{V}_k(x, y)|_{(x_0, y_0)} = g_k(x_0, y_0), k = 1, 2, \quad \forall (x_0, y_0) \in \partial D, \quad (17)$$

where $g_k: \partial D \longrightarrow \mathbb{R}$, $k = 1, 2$, are given functions.

We shall call this problem as the (u_x, v_y) -problem. Let $W: D \longrightarrow \mathbb{R}$ be an unknown biharmonic function. Further we mean by this function the Airy stress function. Denote

$$C_k[W](x, y) := W_k(x, y) + \kappa_0 W_0(x, y), k = 1, 2, \quad \forall (x, y) \in \text{cl}D, \quad (18)$$

where

$$W_0(x, y) := \Delta_2 W(x, y), \quad (19)$$

$$W_1(x, y) := \frac{\partial^2 W(x, y)}{\partial x^2}, W_2(x, y) := \frac{\partial^2 W(x, y)}{\partial y^2}, \quad (20)$$

$\kappa_0 := \frac{\lambda + 2\mu}{2(\lambda + \mu)}$ and μ, λ are Lamé constants (cf., e.g., [8, p. 2]).

The following equalities are valid in D (cf., e.g., [8, pp. 8 – 9], [9, p. 5]):

$$2\mu \mathcal{V}_k(x, y) = C_k[W](x, y), k = 1, 2, \quad \forall (x, y) \in D. \quad (21)$$

Then solving the (u_x, v_y) -problem is equivalent to finding in D quantities of $C_k[W]$, $k = 1, 2$, where boundary values of an unknown biharmonic function $W: D \longrightarrow \mathbb{R}$ satisfy the system

$$2\mu C_k[W](x_0, y_0) = g_k(x_0, y_0), k = 1, 2, \quad \forall (x_0, y_0) \in \partial D. \quad (22)$$

Theorem 2. *The (u_x, v_y) -problem is equivalent to boundary value problem on finding in D the second derivatives $\frac{\partial^2 W(x, y)}{\partial x^2}$, $\frac{\partial^2 W(x, y)}{\partial y^2}$ of a biharmonic function $W \in \mathcal{C}^2(\text{cl}D)$, which satisfy for all $(x_0, y_0) \in \partial D$ the boundary data:*

$$\frac{\partial^2 W(x, y)}{\partial x^2} \Big|_{(x_0, y_0)} = \lambda g_1(x_0, y_0) + (\lambda + 2\mu) g_2(x_0, y_0), \quad (23)$$

$$\frac{\partial^2 W(x, y)}{\partial y^2} \Big|_{(x_0, y_0)} = (\lambda + 2\mu) g_1(x_0, y_0) + \lambda g_2(x_0, y_0). \quad (24)$$

Then a general solution of the (u_x, v_y) -problem is expressed by the formulas:

$$\mathcal{V}_k(x, y) = \frac{1}{2\mu} C_k[W](x, y), k = 1, 2, \quad \forall (x, y) \in D. \quad (25)$$

Proof. Adding equalities of the system (21) term by term, taking into account definitions (18) and the value of κ_0 , we get

$$2\mu (\mathcal{V}_1(x, y) + \mathcal{V}_2(x, y)) = -\frac{\lambda}{2(\lambda + \mu)} \Delta_2 W(x, y) \quad \forall (x, y) \in D. \quad (26)$$

Now, it follows from formulas (21) and (26), definitions (18), inclusions $\mathcal{V}_k \in \mathcal{C}(\text{cl}D)$, $k = 1, 2$, that functions $\Delta_2 W$, $\frac{\partial^2 W}{\partial x^2}$, $\frac{\partial^2 W}{\partial y^2}$ belong to $\mathcal{C}(\text{cl}D)$, therefore, $W \in \mathcal{C}^2(\text{cl}D)$.

Now, solving the system (22) with respect to $\frac{\partial^2 W(x, y)}{\partial x^2} \Big|_{(x_0, y_0)}$ and $\frac{\partial^2 W(x, y)}{\partial y^2} \Big|_{(x_0, y_0)}$, we obtain the system of equations (23), (24). Formulas (25) are followed the equalities (21).

In a similar way, we can prove that solving of a boundary value problem for biharmonic function W with boundary data (23), (23) implies a solution of the (u_x, v_y) -problem by the formulas (25), where a function (19) is a sum of functions in (20). The theorem is proved.

Theorem 3. *A general solution of the homogeneous (u_x, v_y) -problem with zero data $g_1 = g_2 \equiv 0$ is the trivial:*

$$\mathcal{V}_k(x, y) \equiv 0, k = 1, 2, \quad \forall (x, y) \in D. \quad (27)$$

Proof. Passing in (26) to a limit, as (x, y) tends to arbitrary fixed point $(x_0, y_0) \in \partial D$, and taking into account definitions (19), we conclude that $W_0(x, y)|_{(x_0, y_0)} = 0$. Inasmuch as, (19) is a harmonic function then from the last boundary equality, we obtain by the maximum principle the equality

$$W_0(x, y) = 0 \quad \forall (x, y) \in \text{cl}D. \quad (28)$$

The equality (28) yields that functions (20) are harmonic in D and, by formulas (23) and (24), vanish on the boundary ∂D , thus, in view of the maximum principle, we have that

$$W_1(x, y) = W_2(x, y) \equiv 0 \quad \forall (x, y) \in D. \quad (29)$$

Substituting in series expressions (18), equalities (28) and (29) into the formulas (25), we get the formulas (27). The theorem is proved.

Note, that in [10, 11] considered expressions of solutions the Lamé equilibrium system in displacements via components U_k , $k = \overline{1, 4}$, of a monogenic function (3). Moreover, the following statement is proved in [11, Theorem 1]:

Let a function (3) is monogenic in a domain D_ζ . Then the next pairs of functions

$$\begin{aligned} u(x, y) &= \frac{2}{\gamma} U_1(x, y) - \frac{2+\gamma}{\gamma} U_4(x, y), \quad v(x, y) = U_2(x, y); \\ u(x, y) &= -\frac{2+\gamma}{\gamma} U_2(x, y) - \frac{2(1+\gamma)}{\gamma} U_3(x, y), \quad v(x, y) = U_4(x, y); \end{aligned}$$

$$u(x, y) = -\frac{2}{\gamma} U_2(x, y) - \frac{2+\gamma}{\gamma} U_3(x, y), \quad v(x, y) = U_1(x, y)$$

are solutions the Lamé equilibrium system in displacements

$$\begin{cases} \Delta u + \gamma \frac{\partial \theta}{\partial x} = 0, \\ \Delta v + \gamma \frac{\partial \theta}{\partial y} = 0, \end{cases} \quad (30)$$

where $\theta := \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, $\gamma := (\lambda + \mu)\mu^{-1}$.

Note, that this statement is generalization of a result in [10] to a general bounded domain D_ζ . Another results of [10] are also generalized to this case.

4 Boundary value (1-4)-problem for monogenic functions

V.F. Kovalev [12, 13] posed the *biharmonic Schwarz type problems* on finding $\Phi \in \mathcal{M}_{\mathbb{B}}(D_\zeta) \cap \mathcal{C}(\text{cl}D_\zeta)$ by given boundary values of U_k and U_m , $1 \leq k < m \leq 4$, in (3) :

$$U_k(x, y) = u_k(\zeta), \quad U_m(x, y) = u_m(\zeta) \quad \forall \zeta \in \partial D_\zeta,$$

where u_k and u_m are given real-valued functions. We shall call this problem by the $(k-m)$ -problem

Some relations between the (1-3)-problem and problems of the theory of elasticity are described in [12, 14–16]. In particular, it is shown that the main biharmonic problem (cf., e.g., [17, p. 194] and [9, p. 13]) on finding a biharmonic function $U: D \rightarrow \mathbb{R}$ with given limiting values of its partial derivatives $\frac{\partial U}{\partial x}|_{(x_0, y_0)}$ and $\frac{\partial U}{\partial y}|_{(x_0, y_0)}$ can be reduced to the (1-3)-problem.

In [13, 18], we investigated the (1-3)-problem for cases where D_ζ is either an upper half-plane or a unit disk in the biharmonic plane. Its solutions were found in explicit forms with using of some integrals analogous to the classic Schwarz integral. Moreover, the (1-3)-problem is solvable unconditionally for a half-plane but it is solvable for a disk if and only if a certain condition is satisfied.

In [14], a certain scheme was proposed for reducing the (1-3)-problem in a simply connected domain with sufficiently smooth boundary to a suitable boundary value problem in a disk with using power series and conformal mappings in the complex plane.

Under general suitable smooth conditions for a boundary of a bounded domain D and using a hypercomplex analog of the Cauchy type integral, we reduce the (1-3) boundary value problem to a system of integral equations on the real axes and establish sufficient conditions under which this system has the Fredholm property [15, 16].

In this section we are interested in the (1-4)-problem with boundary data

$$U_k[\Phi(\zeta_0)] = u_k(\zeta_0), \quad k \in \{1, 4\}, \quad \forall \zeta_0 = x_0 e_1 + y_0 e_2 \in \partial D_\zeta. \quad (31)$$

Theorem 4. *Let W is a general biharmonic function from $\mathcal{C}^2(\text{cl}D)$ satisfying the boundary conditions (22). Then W rebuilds a general solution of the (u_x, v_y) -problem with boundary data (17) by the formulas (21).*

A general solution Φ of the (1-4)-problem with boundary data

$$u_1 = \lambda g_1 + (\lambda + 2\mu) g_2, u_4 = -\mu g_1 + \mu g_2, \quad (32)$$

generate in D the second order derivatives $\frac{\partial^2 W}{\partial x^2}, \frac{\partial^2 W}{\partial y^2}$ by the formulas (10). A general solution of the (u_x, v_y) -problem for any $(x, y) \in \text{cl}D$ is expressed by the equalities

$$2\mu \frac{\partial u(x, y)}{\partial x} = \frac{\mu}{\lambda + \mu} U_1 [\Phi(\zeta)] - \frac{\lambda + 2\mu}{\lambda + \mu} U_4 [\Phi(\zeta)], \quad (33)$$

$$2\mu \frac{\partial v(x, y)}{\partial y} = \frac{\mu}{\lambda + \mu} U_1 [\Phi(\zeta)] + \frac{\lambda + 2\mu}{\lambda + \mu} U_4 [\Phi(\zeta)], \quad (34)$$

where $\zeta = xe_1 + ye_2 \in \text{cl}D_\zeta$.

Proof. By Theorem 2, the (u_x, v_y) -problem is equivalent to finding the second order partial derivatives $\frac{\partial^2 W}{\partial x^2}, \frac{\partial^2 W}{\partial y^2}$ of a sought-for biharmonic function $W \in \mathcal{C}^2(\text{cl}D)$, which satisfy the limiting conditions (23) and (24).

Let Φ be a required general solution of the (1-4)-problem with boundary data (32). Then, taking into account that D is a simply-connected, obtain that there exists a function $\Phi_* \in \mathcal{M}_{\mathbb{B}}(D_\zeta)$ such that $\Phi(\zeta) = \Phi''_*(\zeta)$ for all $\zeta \in D_\zeta$ and $U_1[\Phi_*] = W$. Then by Theorem 1 the relations (10) are true, substitute them into the expressions (18).

After computations, we obtain that $C_1[W]$ is equal to the right-part in (33), $C_2[W]$ is equal to the right-part in (34). Passing to a limit in these equalities, as (x, y) tends to arbitrary boundary point $(x_0, y_0) \in \partial D_\zeta$, we get with use of equalities (22), (32) an equatily.

Substituting into the right-part in (21) formulas (10), we obtain the equalities (33), (34). The theorem is proved.

5 (u_x, v_y) -problem and the elastic equilibrium

We want to find how a solution of the (u_x, v_y) -problem generates stresses $\sigma_x, \tau_{xy}, \sigma_y$. Assume that we know a general solution of the (u_x, v_y) -problem $\mathcal{V}_1 = \frac{\partial u}{\partial x}, \mathcal{V}_2 = \frac{\partial v}{\partial y}$. Then an unknown the Airy stress function W need to satisfy conditions (21). By the generalized Hooke's law, we have the system of three equations:

$$\sigma_x = (\lambda + 2\mu) \frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y}, \quad (35)$$

$$\sigma_y = \lambda \frac{\partial u}{\partial x} + (\lambda + 2\mu) \frac{\partial v}{\partial y}, \quad (36)$$

$$\tau_{xy} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right). \quad (37)$$

Thus, stresses σ_x and σ_y are found in D by the formulas (35), (36), where $\mathcal{V}_1 = \frac{\partial u}{\partial x}$, $\mathcal{V}_2 = \frac{\partial v}{\partial y}$. Values of a stress τ_{xy} can not be found without values of the second order partial derivatives (20) in the domain D , but the latter can be found with use of Theorem 4 and taking into account a equivalence of the (u_x, v_y) -problem and the appropriate (1-4)-problem.

Indeed, it is well-know (cf., e.g., [8, p.,6]), that the right-hand side of the equality (37) equals $-\frac{\partial^2 W}{\partial x \partial y}$, consequently, $\tau_{xy} = -\frac{\partial^2 W}{\partial x \partial y}$. Therefore, a process of finding a stress τ_{xy} is reduced to finding the mixed second order partial derivative $W_{1,1} := \frac{\partial^2 W}{\partial x \partial y}$ in the domain D . After solving the (1-4)-problem with boundary data (32), we rebuild in D functions W_k , $k = 1, 2$, in (20). We have equalities

$$\tau_{xy} \equiv \tau_{xy}(x, y) = -W_{1,1}(x, y) = - \int_{(x_*, y_*)}^{(x, y)} \frac{\partial W_1}{\partial x} dx - \int_{(x_*, y_*)}^{(x, y)} \frac{\partial W_2}{\partial y} dy \quad \forall (x, y) \in D, \quad (38)$$

where (x_*, y_*) is a fixed point in D , integration means along any piecewise smooth curve, which joints this point with a point with variable coordinates $(x, y) \in D$.

Consequently, formulas (35), (36), (38) deliver stresses under boundary conditions (17).

In order to find displacements u and v we need to obtain their partial derivatives $\mathcal{V}_3 = \frac{\partial u}{\partial y}$, $\mathcal{V}_4 = \frac{\partial v}{\partial x}$ in the domain D , so that, displacements are obtained by the formulas

$$u \equiv u(x, y) = \int_{(x_*, y_*)}^{(x, y)} \mathcal{V}_1 dx + \mathcal{V}_3 dy, \quad v \equiv v(x, y) = \int_{(x_*, y_*)}^{(x, y)} \mathcal{V}_4 dx + \mathcal{V}_2 dy \quad \forall (x, y) \in D. \quad (39)$$

Let us find functions \mathcal{V}_3 and \mathcal{V}_4 . The following formulas are fulfilled in D (cf., e.g., [9, p. 6], [8, p. 9] with $f_1 = f_2 \equiv 0$, $W := U$):

$$2\mu u = -\frac{\partial W}{\partial x} + \kappa_0(4p), \quad 2\mu v = -\frac{\partial W}{\partial y} + \kappa_0(4q),$$

where $W_0 + i\widetilde{W}_0 = 4\varphi'(z)$, \widetilde{W}_0 is a harmonic conjugate of W_0 in D , $\varphi(z) = p(x, y) + iq(x, y)$ is an analytic function of the variable $z = x + iy$ in the domain D_z . These formulas together with the Cauchy–Riemann conditions for analytic function $\varphi = p + iq$ implies the equalities

$$2\mu \mathcal{V}_3 = -W_{1,1} - \kappa_0 \widetilde{W}_0, \quad 2\mu \mathcal{V}_4 = -W_{1,1} + \kappa_0 \widetilde{W}_0.$$

A function $W_{1,1}$ is found from (38), \widetilde{W}_0 is obtained from the Cauchy–Riemann conditions for analytic function $\varphi = p + iq$.

Finally, formulas (35), (36), (38), (39) define the sought-for elastic equilibrium.

6 Monogenic functions approaches for displacements in 3-D Elasticity

Approaches of monogenic functions with values in non-commutative real algebras are using and developing in 3-D Elasticity. For investigation of the space analog of the equilibrium Lamé system (30), considered monogenic (in the sense of some analog of (4)) functions $f: \mathcal{A} \cong \mathbb{R}^3 \longrightarrow \tilde{\mathbb{H}}$, where $\tilde{\mathbb{H}} \equiv \mathbb{H} := x_0 + x_1e_1 + x_2e_2 + x_3e_3$, $x_k \in \mathbb{R}$, $k = \overline{0, 4}$, ($e_ke_j + e_je_k = -2\delta_{k,j}$, $k, j = 1, 2, 3$; $e_1e_2 = e_3$) are real quaternions (cf., e.g., [19–21]) or $\tilde{\mathbb{H}} \equiv \mathcal{A} := x_0 + x_1e_1 + x_2e_2 \subset \mathbb{H}$ (cf., e.g., [22–24]). In this way, a general regular solution of the noticed system for a domain with some convex structure is expressed as a sum of such hypercomplex monogenic functions (for instance, in [24]: a sum of (paravector-valued) monogenic, an anti-monogenic and a ψ -hyperholomorphic functions, respectively). So, this expressions are alternative to the Kolosov-Muskhelishvili formula for the elastic displacement field and they can be used to solving boundary value problems for monogenic functions.

References

- [1] Kovalev V. F., Mel'nichenko I. P. Biharmonic functions on the biharmonic plane, *Reports Acad. Sci. USSR, ser. A.*, No. 8 (1981), 25–27 (in Russian).
- [2] Mel'nichenko I. P. Biharmonic bases in algebras of the second rank, *Ukr. Math. J.*, **38** (1986), No. 2, 252–254.
- [3] Grishchuk S. V., Plaksa S. A. Monogenic functions in a biharmonic algebra, *Ukr. Mat. Zh.*, **61** (2009), No. 12, 1587–1596 (in Russian); English transl. (Springer) in *Ukr. Math. J.*, **61** (2009), No. 12, 1865–1876.
- [4] Plaksa S. A., Gryshchuk S. V., Shpakivskyi V. S. Commutative algebras of monogenic functions associated with classic equations of mathematical physics, in “*Complex Analysis and Dynamical Systems IV*”, *Contemporary Mathematics*, **553** (2011), Amer. Math. Soc., Providence, RI, 245–258.
- [5] Gryshchuk S. V., Plaksa S. A. Basic Properties of Monogenic Functions in a Biharmonic Plane, in: “*Complex Analysis and Dynamical Systems V*”, *Contemporary Mathematics*, **591** (2013), Amer. Math. Soc., Providence, RI, 127–134.
- [6] Plaksa S. A. Commutative Algebras Associated with Classic Equations of Mathematical Physics, in: *Advances in Applied Analysis / Trends in Mathematics*/, Springer Basel AG (2012), 177–223.
- [7] Gryshchuk S. V., Plaksa S. A. Monogenic functions in a biharmonic plane, *Reports Acad. Sci. Ukraine /Mat. Pryr. Tekh. Nauky*/, No. 12 (2009), 13–20 (in Russian).
- [8] Lu J. *Complex Variable Methods in Plane Elasticity /Series in Pure Mathematics*, **22**/, World Scientific, Singapore, 1995.
- [9] Mikhlin S. G. *The plane problem of the theory of elasticity*, *Trans. Inst. of seismology, Acad. Sci. USSR*, No. 65, Acad. Sci. USSR Publ. House, Moscow–Leningrad, 1935 (in Russian).
- [10] Kovalev V. F., Mel'nichenko I. P. Biharmonic potentials and plane isotropic displacement fields, *Ukr. Mat. Zh.*, **40** (1988), No. 2, 229–231 (in Russian); English transl. (Springer) in *Ukr. Math. J.*, **40** (1988), No. 2, 197–199.
- [11] Gryshchuk S. V. Hypercomplex monogenic functions of the biharmonic variable in some problems of the plane elasticity, *Reports Acad. Sci. USSR, ser. A.*, No. 6 (2015), 7–12 (in Ukrainian).
- [12] Kovalev V. F. Biharmonic Schwarz problem, Preprint No. 86.16, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, (1986) (in Russian).
- [13] Gryshchuk S. V., Plaksa S. A. Schwartz-type integrals in a biharmonic plane, *Intern. J. of Pure and Appl. Math.*, **83** (2013), No. 1, 193–211.

- [14] Gryshchuk S. V. Power series and conformal mappings in one boundary value problem for monogenic functions of the biharmonic variable, *Zb. Pr. Inst. Mat. NAN Ukraine*, **11** (2014), No. 1, 93–107.
- [15] Gryshchuk S. V., Plaksa S. A. Monogenic functions in the biharmonic boundary value problem, *ArXiv preprint* / arXiv:1505.02518v1 [math.AP] /, (2015).
- [16] Gryshchuk S. V., Plaksa S. A. Monogenic functions in the biharmonic boundary value problem, *Math. Meth. in the Appl. Sci.*, in a publishing process: published online in Wiley Online Library (2015), DOI:10.1002/mma.3741.
- [17] Smirnov V. I. *Course of Higher Mathematics. Vol. 3, Part 2*, Pergamon Press, Oxford, 1964.
- [18] Gryshchuk S. V., Plaksa S. A. Biharmonic Schwartz integral for a half-plane, in: “*Progress in Analysis*”: Proc. of the 8th Congress of the International Society for Analysis, its Applications, and Computation (ISAAC), Moscow, Russia, 22 – 27 August, 2011, Conf. Proc., **1**, People’s Friendship University of Russia, Moscow (2012), 93–99.
- [19] Grigorev Yu. Three-dimensional Quaternionic Analogue of the KolosovMuskhelishvili Formulae, in: *Hypercomplex Analysis: New Perspectives and Applications /Trends in Mathematics/*, Bernstein S., Kählen, Sabadini I., Sommen F. (eds.), Springer International Publishing, Switzerland, (2014), 145 – 166.
- [20] Grigorev Yu. Regular quaternionic functions and their applications, 20th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering, Weimar, Germany, 20-22 July 2015, Proceedings of the IKM 2015, K. Gürlebeck and T. Lahmer (eds.), Germany, Weimar, ISSN 1611-4086 (2015), 45 – 50.
- [21] Bock S., Gürlebeck K. On a spatial generalization of the Kolosov-Muskhelishvili formulae, *Math. Meth. in the Applied Sciences*, **32** (2009), No. 2, 223-240.
- [22] Gürlebeck K., Nguyen H. M. ψ -hyperholomorphic functions and an application to elasticity problems, Numerical analysis and applied mathematics. International conference of numerical analysis and applied mathematics (ICNAAM 2014), Rhodes, Greece, 22 - 28th September 2014, AIP Conference Proceedings vol. **1648**, Edited by Theodore E. Simos and Charalambos Tsitouras, (2015); doi: 10.1063/1.4912656.
- [23] Weisz-Patrault D., Sebastian B., Gürlebeck K. Three-dimensional elasticity based on quaternion-valued potentials, *Intern. J. of Solids and Structures*, **51** (2014), No.19, 3422–3430.
- [24] Bock S., Gürlebeck K., Legatiuk D. and Nguyen H. M. ψ -Hyperholomorphic functions and a Kolosov–Muskhelishvili formula, *Math. Meth. in the Appl. Sci.*, Mart 12, 2015, dx.doi.org/10.1002/mma.3431.